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THE UNIVERSITY OF CHICAGO
Founded by John D. Rockefeller

On the
Reduction of hyperelliptic Integrals ($p=3$) to elliptic Integrals by transformation of the second and third Degrees.

A DISSERTATION
submitted to the Faculties of the Graduate Schools of Arts, Literature, and Science, in candidacy for the degree of
Doctor of Philosophy in Mathematics

by

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Introduction.

The principal subject of this paper is an application of the cubic involution to the problem of reduction to elliptic integrals, of hyperelliptic integrals of genus $p = 3$ and of the first kind, by a rational transformation of the third degree.

It forms thus a continuation of Professor **Bolza**'s researches on the cubic transformations of elliptic integrals, and cubic reductions of hyperelliptic integrals of genus $p = 2$. [1]

It is also closely connected with the work of **J. I. Hutchinson**, in his dissertation[2] where be applies the quadratic involution to a similar problem where $p = 2$.

In Part I the general case of cubic reduction is considered, after the connection of the problem with the theory of cubic involution has been established (§ 1).

The invariant condition, which must be satisfied, in the case of cubic reduction, by the octavic on whose square root the hyperelliptic integral depends, is obtained (§ 2), and the most general reducible integral is determined in its most general form, first in a normal form (§ 3), then independent of any normal form (§ 4 — § 7.)

The main subject of Part II is the singular case of simultaneous reduction of the second and third degree, that is, where two hyperelliptic integrals having the same octavic are reducible to elliptic integrals, one by a transformation of the second degree, and the other by one of the third (§ 3 — § 8).

As an introduction to this part the general case of quadratic reductions ($p = 3$) is considered (§ 1), and the special cases of more than one reduction for a given radical are determined. A complete table of such cases is formed (§ 2).

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[1] "Die cubische Involution und die Dreitheilung und Transformation dritter Ordnung der elliptischen Functionen" and "Zur Reduction hyperelliptischer Integrale erster Ordnung, auf elliptische mittels einer Transformation dritten Grades". Math. Ann. Bd. 50, pag. 68 and pag. 314.

[2] "On the reduction of Hyperelliptic Functions ($p = 2$) to Elliptic Functions by a transformation of the second degree." Chicago 1897.

In the last Part, the involution is specialized to one containing two cubes. For this special form the most general reducible integral is determined (§ 1), and the cases of simultaneous reduction of the second and third degree, (§ 2).

For one of the latter cases, in which the hyperelliptic curve can be reduced to the normal form

$$y^2 = x\,(1 - x^6),$$

the reduction problem is studied by means of Weierstrass-Picard's theorem on the periods of reducible integrals (§ 3). This leads us, not only to a confirmation of the results previously obtained by algebraic methods, but also to a determination of all hyperelliptic integrals of the first kind, belonging to the curve

$$y^2 = x\,(1 - x^6),$$

which are reducible to elliptic integrals by rational transformations of any degree.

Part I.

§ I. Connection of the problem with the cubic involution.

We will first investigate the general form of a hyperelliptic integral of the first kind and of genus three, which is reducible to an elliptic integral by a rational transformation of the third degree.

The method used by Jacobi for the transformation of elliptic integrals[1]), can be extended at once to the investigation of the reducibility of hyperelliptic integrals, and furnishes the results: —

In order that the hyperelliptic integral

$$\int \frac{q(x_1\, x_2)\,(xdx)}{\sqrt{R\,(x_1\, x_2)}}$$

where $q\ (x_1\ x_2)$ is a quadratic and $R\,(x_1\ x_2)$ an octavic quantic, shall be reducible to the elliptic integral

$$M \int \frac{(ydy)}{\sqrt{(y_1 - \lambda_1\, y_2)\,(y_1 - \lambda_2\, y_2)\,(y_1 - \lambda_3\, y_2)\,(y_1 - \lambda_4\, y_2)}}$$

M being a constant, by the transformation

$$\begin{aligned} y_1 &= U \\ y_2 &= V, \end{aligned}$$

U and V being cubic quantics in $(x_1\ x_2)$, it is necessary and sufficient firstly, that R may be broken up into two cubic factors R_1, R_2, and two linear factors $(x e_1)$, $(x e_2)$, such that

$$\begin{cases} U - \lambda_1\, V = R_1 \\ U - \lambda_2\, V = R_2 \\ U - \lambda_3\, V = (xd_1)^2\ (xe_1) \\ U - \lambda_4\, V = (xd_2)^2\ (xe_2) \end{cases}$$

and therefore $R = R_1 R_2\ (xe_1)\ (xe_2)$.

Evidently, the four factors $U - \lambda i\ V$ belong to a cubic involution, and the last two are branch triples of that involution, and therefore, this condition requires that the octavic shall be decomposable into three factors, two cubics which shall define a cubic involution and a quadratic whose roots are branch points of that involution.

---

[1]) Fundamenta Nova, Werke, Bd. I, pag. 55.

And secondly, since $(ydy) = UdV - VdU$ which, being proportion to the Jacobian of U and V, has for roots, the four double points the involution, and since two of these double points appeared in square factors in the quadratics (3), the quadratic $q(x_1 x_2)$ must have as roots, the two double points of the involution, not corresponding to the branch points occurring in $R\ (x_1 x_2)$.

Thus a hyperelliptic integral which is reducible, must have the form

(4) $$\int \frac{(xd_3)\,(xd_4)\,(xdx)}{\sqrt{(xe_1)\,(xe_2)\,(\lambda_1 f_1 + \lambda_2 f_2)\,(\mu_1 f_1 + \mu_2 f_2)}}$$

where f_1 and f_2 are linear homogeneous functions of the cubics U and V and $(xd_i)^2\ (xe_i)$ for $i = 1\ 2\ 3\ 4$, denote the branch triples of the involution

$$U - \lambda V;$$

and if an integral have this form, it is reducible.

§ 2. The cubic involution and the invariant condition for reducibility.

We collect in this paragraph, those properties[1]) of the cubic involution, of which we will make frequent use.

A cubic involution is defined by the equation

$$f_1 + \lambda . f_2 = 0$$

where f_1 and f_2 are two given cubic quantics, and λ is an arbitrary parameter.

We require the following combinants,

(5) $$\vartheta = (f_1 f_2)_1\ ;\ J = (f_1 f_2)_3\ ;\ H_\vartheta = (\vartheta\vartheta)_2\ ;\ i_\vartheta = (\vartheta\vartheta)_4\ ;\ j_\vartheta = (\vartheta H)_4$$

Being given one point y of a triple, the other two points are the two roots of the "affinity equation"

(6) $$3\ \vartheta_x^2 \vartheta_y^2 + \tfrac{1}{2}\ J\ (xy)^2 = 0.$$

The double point of the involution are the roots of the equation

(7) $$\vartheta = 0,$$

and the four branch points, of the equation

(8) $$3\ H_\vartheta + J\vartheta = 0.$$

We also define a biquadratic Γ as follows: —

(9) $$\Gamma = 2\ H_\vartheta + \tfrac{1}{3}\ J\vartheta$$

---

[1]) We refer for these statements to Bolza's paper in the Annalen, Bd. I, § 1, where the literature of the subject is given.

The first polar of Γ with respect to an arbitrary constant λ is the cul involution itself.

For a given ϑ there exists besides the involution of which we ha just spoken, another called the conjugate involution.

For this involution, the combinant J has the opposite sign, i. $\bar{J} = -J$.

The rest of the combinants in (5) are unchanged. The affinity, ar branch equations, are obtained form (6) and (8) by replacing J by — while the corresponding $\bar{\Gamma}$ is so defined: —

$$\bar{\Gamma} = 2\,H_\vartheta - \tfrac{1}{3}\,J\vartheta.$$

We saw, in § 1 that $R\ (x_1 x_2)$ must, for a reducible integral, hav the form $R_1 R_2\ (xe_1)\ (xe_2)$.

Denoting the quadratic whose roots are e_1 and e_2 by N, we hav the condition for reducibility in the following form: —

Theorem. In order that there shall exist a reducibl hyperelliptic integral of the first kind belonging to $R\ (x_1 x_2)$ it is necessary and sufficient that $R\ (x_1 x_2)$ be decomposabl into the factors R_1, R_2, and N, such that the quadratic N shall divide exactly the branch form $3H_\vartheta + J\vartheta$, for th involution determined by the two cubic factors R_1 and R_2.

This condition is equivalent to two algebraic relations between the roots of R, in accordance with the general theory of reduction[1]); it can be represented by the identical vanishing of a simultaneous biquadratic covariant of N and $3\,H + J\vartheta$.[2])

§ 3. A normal form for the reducible integral.

From the form of the reducible integral in § 1, we see that the double points are associated in pairs, and for that reason we choose the following normal form, which Klein uses in his Modulfunctionen[3]).

Denoting the roots of ϑ by

$$x_1^{(i)}, x_2^{(i)}: i = 1\ 2\ 3\ 4, \text{ and } x_1^{(i)} x_2^{(k)} - x_2^{(i)} x_1^{(k)} \text{ by } (ik),$$

we have the three invariants

$$\begin{aligned} A &= (12)\ (34)\ ; \\ B &= (13)\ (42)\ ; \\ C &= (14)\ (23)\ ; \end{aligned}$$

where $A + B + C = 0$.

Then ϑ can be transformed by a linear transformation of determinant $= 1$ into $\vartheta = x_1 x_2\ (x_2 - x_1)\ (Ax_1 + Bx_2)$.

---

[1]) Königsberger, Crelle 85.

[2]) Clebsch, Binäre Formen § 27, pag. 94.

[3]) Modulfunctionen pag. 7.

Here we have the pairing of two roots put in evidence.

We find the branch points from the affinity equation. Forming this equation for any double point, that point will be a root of it, but the other root is the corresponding branch point. The affinity equation corresponding to the point d, is given symbolically thus, —

$$6\,\vartheta_x^2\,\vartheta_d^2 + J\,(xd)^2 = 0.$$

This must be expressed in terms of the coefficients of our normal form of ϑ and also x and d.

The affinity equation becomes

$$\begin{aligned} &- 3A(x_1^2 d_1 d_2 + x_1 x_2 d_1^2) + (A-B)(4x_1 x_2 d_1 d_2 + x_1^2 d_2^2 + x_2^2 d_1^2) \\ &+ 3B(x_1 x_2 d_2^2 + x_2 d_1 d_2) + J(x_1^2 d_2^2 - 2x_1 x_2 d_1 d_2 + x_2^2 d_1^2) = 0. \end{aligned} \tag{11}$$

The invariants can be expressed in the following way [1]).

$$A = 2i\sqrt{3}\,l_1\,\overline{l_1}\ ;\ B = 2i\sqrt{3}\,l_2\,\overline{l_2}\ ;\ l = 2i\sqrt{3}\,l_3\,\overline{l_3}\ ;$$

where the l's are the irrational invariants of the cubic involution used by Bolza [1]).

They are defined by the equations $l_i = \sqrt{K_i}$ where the K's are roots of the resolvent cubic,

$$4K^3 + 4JK^2 + J^2K - \overline{\Omega} = 0$$

and are connected by the following relations, —

$$\begin{gathered} l_1 + l_2 + l_3 = 0\ ;\ l_1 - l_2 = \sqrt{3}\,\overline{l_3}\ ;\ l_2 - l_3 = i\sqrt{3}\,\overline{l_1}\ ; \\ l_3 - l_1 = i\sqrt{3}\,\overline{l_2}. \end{gathered} \tag{12}$$

Using these values for the invariants the above method yields the following four branch triples:

	double point.	branch point.
I.	$x_1 = 0$	$x_1 = i\sqrt{3}\,\dfrac{\overline{l_2}}{l_1}\,x_2$
II.	$x_2 = 0$	$x_1 = \dfrac{l_2}{i\sqrt{3}\,\overline{l_1}}\,x_2$
III.	$x_1 = x_2$	$x_1 = \dfrac{l_2}{l_1}\,x_2$
IV.	$x_1 = -\dfrac{l_2\,\overline{l_2}}{l_1\,\overline{l_1}}\,x_2$	$x_1 = -\dfrac{\overline{l_2}}{\overline{l_1}}\,x_2\,.$

By the method of § 1 and using the first two double points as the roots of $q\,(x_1\,x_2)$ and the first two branch triples as U and V, we obtain the result:

Theorem: Every hyperelliptic integral which is reducible by a transformation of the third degree, can be transformed, by a linear transformation of the independent variable, into the form —

---

[1]) l. c. 50. (pag. 94) equation (77).

$$2i\sqrt{3}\int\frac{x_1 x_2 (x_1 dx)}{\sqrt{\begin{aligned}&\{(l_1 x_1 + l_2 x_2)\ (\bar{l}_1 x_1 + \bar{l}_2 x_2) \times \\ &[\lambda_1 (l_1 x_1 + i\sqrt{3}\,\bar{l}_2 x_2)\, x_1^2 + \lambda_2 (i\sqrt{3}\,\bar{l}_1 x_1 - l_2 x_2)\, x_2^2] \times \\ &[\mu_1 (l_1 x_1 + i\sqrt{3}\,\bar{l}_2 x_2)\, x_1^2 + \mu_2 (i\sqrt{3}\,\bar{l}_1 x_1 - l_2 x_2)\, x_2^2]\}\end{aligned}}}$$

which by the transformation

$$\begin{aligned}y_1 &= (l_1 x_1 + i\sqrt{3}\,\bar{l}_2 x_2)\, x_1^2 \\ y_2 &= (i\sqrt{3}\,\bar{l}_1 x_1 - l_2 x_2)\, x_2^2\end{aligned}$$

reduces to the elliptic integral

$$\int\frac{(y\,dy)}{\sqrt{(y_1 + y_2)\ (l_1 \bar{l}_1^3 y_1 - l_2 \bar{l}_2^3 y_2)\ (\lambda_1 y_1 + \lambda_2 y_2)\ (\mu_1 y_1 + \mu_2 y_2)}}$$

where $l_1\, l_2\, l_3$ are three parameters satisfying the relation $l_1 + l_2 + l_3 = 0$, while $\bar{l}_1$, $\bar{l}_2$, and $\bar{l}_3$ are determined by (12); the parameters $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$, are arbitrary. The integral thus depends on three independent non-homogeneous parameters, which agrees with the above result in the end of § 2.

§ 4. A Theorem on cubic involutions.

We proceed now to give the reducible integral in its most general form independent of any normal form. For this purpose, we add a lemma on cubic involutions, which will be useful since, in our problem, the double points and also the branch points, are associated in pairs.

We have immediately from Clebsch, Binäre Formen pag. 159:—

$$\vartheta = \frac{2}{m_2 - m_3}(\chi^2 - \psi^2) \tag{14}$$

where ψ and χ are two of the quadratic factors of T_ϑ, and the m's are roots of the cubic resolvent of ϑ.

We deduce: —

$$3 H_\vartheta + J\vartheta = \frac{6}{m_2 - m_3}(l_3^2 \psi^2 - l_2^2 \chi^2) \tag{15}$$

$$\Gamma = 2H + \tfrac{1}{3} J\vartheta = \frac{4\bar{l}_1}{m_2 - m_3}(\bar{l}_2 \psi^2 - \bar{l}_3 \chi^2) \tag{16}$$

From inspection it seems highly probable that the factor $(\psi - \chi)$ of (14), should yield the double points which correspond to the branch points given by one of the factors $(l_3 \psi \pm l_2 \chi)$ of (15).

We will proceed to find out if this surmise be true.

For our work we require the affinity equation

$$3\,\vartheta_x^2 \vartheta_y^2 + \tfrac{1}{2} J\,(xy)^2 = 0 \tag{6}$$

expressed in terms of ψ, χ and the m's.

First find $\vartheta_x^2 \vartheta_y^2$; using the expression of ϑ given in (14) we have the second polar of χ^2 with regard to $y = \chi(x)\,\chi(y) - \frac{1}{3}\,(xy)^2\,A_{22}$

and „ „ „ „ ψ^2 „ „ „ $y = \psi\, x)\,\psi(y) - \frac{1}{3}\,(xy)\,A_{11}$

where [1])

$$(17)\qquad \begin{cases} A_{11} = \frac{1}{2}\,(m_1 - m_2)\,(m_2 - m_3) \\ A_{22} = \frac{1}{2}\,(m_2 - m_3)\,(m_3 - m_1) \end{cases}$$

∴ we find after substituting these values —

$$\vartheta_x^2 \vartheta_y^2 = \frac{2}{m_2 - m_3}\,(\chi(x)\,\chi(y) - \psi(x)\,\psi(y)) + m_1\,(xy)^2$$

whence the affinity equation is

$$(18)\qquad Af(y) \equiv \frac{2}{m_2 - m_3}\,[\chi(x)\,\chi(y) - \psi(x)\,\psi(y)] + l_2\,l_3\,(xy)^2 = 0$$

since $(m_1 + \frac{1}{6}\,J) = \overline{l_2}\,\overline{l_3}$. [2])

(19) Let us suppose $l_3\psi - l_2\chi = (xe)\,(xe')$.

Then the method of testing if our surmise be true, is the following; — we form the affinity equations for e and e'; their product must be a perfect square, and if our surmise be true, the square of one of the factors $(\psi \pm \chi)$.

Now $Af\,(e).Af\,(e')$ is the resultant of (18) and (19).

We form this resultant by means of symbolic notation.

Let $Af(y) = \alpha_y^2 = \beta_y^2$; $\chi = \chi_y^2$; $\psi = \psi_y^2$; $l_3\psi - l_2\chi = a_y^2 = b_y^2$; and $p = (xy)^2 = p_y^2$.

The resultant is

$$(20)\qquad R = (a\alpha)^2\,(b\beta)^2 - (ab)^2\,(\alpha\beta)^2.$$

We find at once, bearing in mind, —

$$(p\psi)_2 = \psi;\ (p\chi)_2 = \chi;\ (\chi\chi)_2 = A_{22};\ (\psi\psi)_2 = A_{11}$$

$$(21)\qquad \text{where}\ A_{11} = \frac{1}{2}\,(m_2 - m_3)\,l_3\,(l_2 - l_1);$$

$$A_{22} = \frac{1}{2}\,(m_2 - m_3)\,l_2\,(l_1 - l_3);$$

that $(a\alpha)^2 = l_1\,(l_3^2\psi - l_2^2\chi)$;

---

[1]) Clebsch, Binäre Formen. § 45.

[2]) Bolza, l. c. (72).

similarly

$$(ab)^2 = \frac{m_2 - m_3}{2} \left[l_3^3 (l_2 - l_1) + l_2^3 (l_1 - l_3)\right] ;$$

$$(\alpha\beta)^2 = \frac{2}{m_2 - m_3} (l_3^2 \psi^2 - l_2^2 \chi^2) ;$$

∴ from (20)

$$R = l_1^2 l_2^2 l_3^2 [\psi - \chi]^2 . \tag{22}$$

∴ our surmise was correct and we have proved the following theorem: —

In any cubic involution the Jacobian and the branch equation, when expressed in terms of the Clebsch functions ψ and χ and the invariants m_s and l_s, assume the forms $\frac{2}{m_2 - m_3} (\chi^2 - \psi^2)$ and $\frac{6}{m_2 - m_3} (l_3^2 \psi^2 - l_2^2 \chi^2)$ respectively; the double points yielded by the factor $(\psi - \chi)$ of the Jacobian correspond to the branch points yielded by the factor $(l_3 \psi - l_2 \chi)$ of the branch equation and there is a like correspondence between the roots of $(\psi + \chi)$ and $(l_3 \psi + l_2 \chi)$.

An entirely analogous reasoning, applied to the conjugate involution, shows that there the double points yielded by $(\psi - \chi)$ of the Jacobian correspond to the branch points yielded by the factor $(\bar{l}_3 \psi - \bar{l}_2 \chi)$; and similarly for the factors $(\psi + \chi)$ and $(\bar{l}_3 \psi + \bar{l}_2 \chi)$.

§ 5. First general form of the reducible integral.

We can now apply the result of the lemma, which has just been proved, to the form of the reducible integral which was given in § 1. (4).

We may suppose: —

$$\begin{gathered} (\psi + \chi) = (xd_1)(xd_2) ; (\psi - \chi) = (xd_3)(xd_4) ; \\ (l_3 \psi + l_2 \chi) = (xe_1)(xe_2) ; (l_3 \psi - l_2 \chi) = (xe_3)(xe_4) ; \\ (\bar{l}_3 \psi + \bar{l}_2 \chi) = (x\bar{e}_1)(x\bar{e}_2) ; (\bar{l}_3 \psi - \bar{l}_2 \chi) = (x\bar{e}_3)(x\bar{e}_4) ; \end{gathered} \tag{23}$$

this however involves an agreement concerning the way in which the homogeneous coordinates of the points d_i e_i $\bar{e}_i$ $(i = 1\ 2\ 3\ 4)$ are chosen.

Also

$$\Gamma = \frac{4i}{\sqrt{3}\, l_1} (\bar{l}_2 \psi^2 - \bar{l}_3 \chi^2) \tag{24}$$

has for its first polar with regard to an arbitrary parameter the cubic involution itself, *i.e* $\Gamma_x^3 \Gamma_\lambda$.

∴ The reducible integral § 1 (4) has the form

$$\int \frac{(\psi - \chi)(x\,dx)}{\sqrt{(l_3 \psi + l_2 \chi)\, \Gamma_x^3 \Gamma_\lambda \Gamma_x'^3 \Gamma_\mu'}} . \tag{25}$$

The transformation

(26) $$y_1 = U \qquad y_2 = V$$

can by a linear transformation of the elliptic integral, be thrown into the form $\Gamma_x^3 \Gamma_y = 0$, which may be expressed thus, —

$$y_1 = \tfrac{1}{4} \frac{\partial \Gamma}{\partial x_2} \quad y_2 = -\tfrac{1}{4} \frac{\partial \Gamma}{\partial x_1}.$$

Let us examine the form of the elliptic integral. Before reduction we have the hyperelliptic in the form

$$\int \frac{\vartheta\,(xdx)}{\sqrt{(xd_1)^2\,(xe_1)\,(xd_2)^2\,(xe_2)\,\Gamma_x^3 \Gamma_\lambda {\Gamma'_x}^3 \Gamma'_\mu}}$$

and by the transformation (26)

(27) $$\begin{aligned} (y\tilde{e}_1) &= \Gamma_x^3 \Gamma_{\tilde{e}_1} = \varrho_1\,(xe_1)\,(xd_1)^2 \quad ^{1)} \\ (y\tilde{e}_2) &= \Gamma_x^3 \Gamma_{\tilde{e}_2} = \varrho_2\,(xe_2)\,(xd_2)^2 \\ (y\lambda) &= \Gamma_x^3 \Gamma_\lambda^2 \\ (y\mu) &= \Gamma_x^3 \Gamma_\mu \end{aligned}$$

$$\therefore\ \varrho\,(xd_1)^2\,(xe_1)\,(xd_2)^2\,(xe_2) = (y\tilde{e}_1)\,y\tilde{e}_2) = (\overline{l}_3\,\psi(y) + \overline{l}_2\,\chi(y))$$

(28) $$\text{and } \Gamma_x^3 \Gamma_\lambda {\Gamma'_x}^3 \Gamma'_\mu = (y\lambda)\,(y\mu) = \text{say } g(y)$$

and we also have

(29) $$(ydy) = \tfrac{3}{2}\,\Omega\,\vartheta\,(xdx) \quad ^{2)}$$

$\therefore$ after reduction our elliptic integral has the form

(30) $$C \int \frac{(ydy)}{\sqrt{[\overline{l}_3\,\psi(y) + \overline{l}_2\,\chi(y)]\;g(y)}}$$

where C is some constant.

The expression $\Gamma_x^3 {\Gamma'_x}^3 \Gamma_\lambda \Gamma'_\mu$ can be thrown into a more serviceable form.

$$\begin{aligned} \Gamma_x^3 {\Gamma'_x}^3 \Gamma_\lambda \Gamma'_\mu &= \tfrac{1}{2} \Gamma_x^3 {\Gamma'_x}^3 (\Gamma_\lambda \Gamma'_\mu + \Gamma'_\lambda \Gamma_\mu) \\ &= \Gamma_x^3 {\Gamma'_x}^3\,(\Gamma g)\,(\Gamma' g') \\ &= \Gamma_\lambda^2 {\Gamma'_x}^2\,(\Gamma_x \Gamma'_x\,\Gamma g\,\Gamma g') \\ &= \tfrac{1}{2}\,\Gamma_x^2 {\Gamma'_x}^2\,\left[(\Gamma g)^2 {\Gamma'_x}^2 + (\Gamma g')^2 \Gamma_x^2 - (\Gamma\Gamma')^2 g_x^2\right] \end{aligned}$$

Clebsch, Bin. Form pg. 40

(31) $$\therefore\ \Gamma_x^3 {\Gamma'_x}^3 \Gamma_\lambda \Gamma'_\mu = [\Gamma g]_2\,\Gamma - \tfrac{1}{2} H_\Gamma g.$$

---

1) Bolza. l. c. (34); 2) Ibid. (38).

∴ The hyperelliptic integral must therefore have the form

$$\int \frac{(\psi^{(x)} - \chi^{(x)})\,(xdx)}{\sqrt{[l_3\,\psi^{(x)} + l_2\,\chi^{(x)}]\,[(\Gamma g)_2\,\Gamma - \tfrac{1}{2} H_{\Gamma} g]}} \tag{32}$$

Equations (26) and (30) give us the transformation and the elliptic integral,— but we have still to find the constant multiplier.

From the theorem § 4 and equation (27), we have

$$\Gamma_x{}^3 \,.\, \Gamma_{\overline{l}_1}\,,\, \Gamma_x{}'^3 \,.\, \Gamma_{\overline{l}_2}{}' = \varrho\,(l_3\,\psi + l_2\,\chi)\,(\psi + \chi)^2 \tag{33}$$

and we will now find this constant ϱ.

We have from equations (23)

$$\left\{\begin{aligned} \psi + \chi &= (xd_1)\,(xd_2)\,; \\ l_3\,\psi + l_2\,\chi &= (xe_1)\,(xe_2)\,; \\ \overline{l}_3\,\psi + \overline{l}_2\,\chi &= (x\bar{e}_1)\,(x\bar{e}_2)\,. \end{aligned}\right. \tag{34}$$

We also know for any quantities λ and μ, that

$$\Gamma_x{}^3\,\Gamma_\lambda\,\Gamma_x{}'^3\,\Gamma_\mu{}' = (g\Gamma)_2\,\Gamma - \tfrac{1}{2}\,H_{\Gamma} g \quad (31)$$

where $g = (x\lambda)\,(x\mu)$;

then

$$\Gamma_x{}^3\,\Gamma_{\bar{e}_1}\,\Gamma_x{}'^3\,\Gamma_{\bar{e}_2}{}' = [[\overline{l}_3\,\psi + \overline{l}_2\,\chi]\,\Gamma]_2\,\Gamma - \tfrac{1}{2}\,H_{\Gamma}\,[\overline{l^3}\,\psi + \overline{l}_2\,\chi]. \tag{35}$$

Comparing this with the equation (33) will determine ϱ.

$$\frac{4i}{\sqrt{3}\,l_1}\,[\overline{l}_3\,\psi + \overline{l}_2\,\chi\,,\,\overline{l}_2\,\psi^2 - l_3\,\chi^2]_2 = [l_3\,\psi + l_2\,\chi\,,\,\Gamma] \quad (24).$$

To evaluate this we find

$$[\psi\,,\psi^2]_2 = \tfrac{2}{3}\,A_{11}\,\psi\,;\;\; [\psi\,,\chi^2] = -\tfrac{1}{3}\,A_{22}\,\psi\,;\;\; [\chi\,,\psi^2]^2 = -\tfrac{1}{3}\,A_{11}\,\chi$$

whence

$$[l_3\,\psi + l_2\,\chi\,,\,\Gamma]_2 = \tfrac{2}{3}\,\overline{l}_2\,\overline{l}_3\,(A_{11}\,\psi - A_{22}\,\chi) + \tfrac{1}{3}\,\overline{l}_3{}^2\,A_{22}\,\psi - \tfrac{1}{3}\,\overline{l}_2{}^2\,A_{11}\,\chi.$$

To find the H_{Γ} we obtain

$$[\psi^2 \,.\, \psi^2]_2 = \tfrac{1}{3}\,A_{11}\,\psi^2\,,\;\; [\psi^2,\chi^2]_2 = -\tfrac{1}{3}\,[A_{22}\,\psi^2 + A_{11}\,\chi^2]$$

and others;

then

$$H_{\Gamma} = \frac{-16}{9\,l_1{}^2}\,[[\overline{l}_2{}^2\,A_{11}\,\psi^2 + \overline{l}_3{}^2\,A_{22}\,\chi^2] + 2\,\overline{l}_2\,\overline{l}_3\,[A_{11}\,\chi^2 + A_{22}\,\psi^2]] \tag{36}$$

and remembering

$$A_{11} = -\tfrac{3}{2}\,l_1\,l_3\,\overline{l}_1\,\overline{l}_3\,;\;\; A_{22} = -\tfrac{3}{2}\,l_1\,l_2\,\overline{l}_1\,\overline{l}_2\,;$$

equation (35) gives us

(37) $$\Gamma x^3\,\Gamma\bar{e}_1\,\Gamma x'^3\,\Gamma\bar{e}_2' = \frac{\Omega}{l_1\,\bar{l}_1}\,(l_3\psi + l_2\chi)\,(\psi + \chi)^2$$

(38) where $\sqrt{\Omega} = 2\;\bar{l}_1\,\bar{l}_2\,\bar{l}_3$ [1]).

Later we will use the conjugate invariant

$$\sqrt{\bar{\Omega}} = 2\;l_1\,l_2\,l_3.$$

We must therefore collect the factors which equations (29) and (37) give us, and also the factor which comes from the ϑ

$$\vartheta = \frac{2}{i\sqrt{3}\,l_1\,\bar{l}_1}\,[\psi^2 - \chi^2] \qquad (14)$$

Theorem: The hyperelliptic integral

(39) $$\frac{\sqrt{3}\quad\Omega^{\frac{1}{2}}}{i\,\sqrt{l_1\,\bar{l}_1}}\int\frac{[\psi(x) - \chi(x)]\;(xdx)}{\sqrt{[l_3\,\psi(x) + l_2\,\chi(x)]\;\left[[g\,\Gamma]_2\,\Gamma - \frac{1}{2}\,H_{\Gamma}g\right]}}$$

by the transformation

$$\varrho y_1 = \frac{1}{4}\,\frac{\partial\Gamma}{\partial x_2};\ \varrho y_2 = -\frac{1}{4}\,\frac{\partial\Gamma}{\partial x_1}$$

reduces to the elliptic integral

$$\int\frac{(ydy)}{\sqrt{[\bar{l}_3\,\psi(y) + \bar{l}_2\,\chi(y)]\,.\,g(y)}}.$$

§ 6. A second general form of the reducible integral.

We can express these integrals and the reduction in terms of the biquadratic Γ, and quantities derived from Γ; so that, knowing Γ, we could find the integral.

From Γ we derive the functions Φ, Ψ, and X in just the same way as φ, ψ, and χ were derived from ϑ; and let n_1, n_2, and n_3 take the place of the invariants m_1, m_2, and m_3 in the work, that is, be the roots of the cubic resolvent for Γ.

The corresponding quantities of $\bar{\Gamma}$, we will denote by, –

$$\bar{n}_1,\ \bar{n}_2,\ \text{and}\ \bar{n}_3.$$

The following relations hold between these n's and cur l's [1]); —

(40)
$$\bar{n}_1 = l_1\sqrt{\bar{\Omega}}, \qquad\qquad n_1 = \bar{l}_1\sqrt{\Omega},$$
$$\bar{n}_2 = l_2\sqrt{\bar{\Omega}}, \quad \text{and similarly} \quad n_2 = \bar{l}_2\sqrt{\Omega},$$
$$\bar{n}_3 = l_2\sqrt{\bar{\Omega}}, \qquad\qquad n_3 = \bar{l}_3\sqrt{\Omega}.$$
$$\bar{n}_1\,.\,\bar{n}_2\,.\,\bar{n}_3 = \tfrac{1}{2}\,\bar{\Omega}^2 \quad ; \quad n_1\,.\,n_2\,.\,n_3 = \tfrac{1}{2}\,\Omega^2$$

and between the n's and $\bar{n}$'s; [2])

---

[1]) Bolza, l. c. 85). [2]) Ibid (86).

$$
\begin{aligned}
i\sqrt{3}\,\sqrt{\frac{\Omega}{\overline{\Omega}}}\,\overline{n}_1 &= n_2 - n_3,\\
i\sqrt{3}\,\sqrt{\frac{\Omega}{\overline{\Omega}}}\,\overline{n}_2 &= n_3 - n_1,\\
i\sqrt{3}\,\sqrt{\frac{\Omega}{\overline{\Omega}}}\,\overline{n}_3 &= n_1 - n_2.
\end{aligned}
\tag{41}
$$

We must now find a relation between these new functions Φ, Ψ, and X and the old functions φ, ψ, and χ.

From Clebsch we know

$$\varphi = H_\vartheta + m_1\,\vartheta,$$

whence we draw

$$
\begin{aligned}
\Phi &= H_\Gamma + n_1\,\Gamma\\
&= \Omega\vartheta + \overline{l}_1\,\Omega^{\frac{1}{2}}\,(2\,H_\vartheta + \tfrac{1}{3}J\vartheta) \qquad {}^{1})\\
&= 2\,\overline{l}_1\,\Omega^{\frac{1}{2}}\,(H_\vartheta + m_1\,\vartheta)
\end{aligned}
$$

remembering that $2\,\overline{l}_1\,\overline{l}_2\,\overline{l}_3 = \sqrt{\Omega}$ (38)

and $\overline{l}_2\,\overline{l}_3 = m_1 - \frac{1}{6}J$ [2]).

Thus we find that

$$
\begin{aligned}
\varphi &= \frac{1}{\sqrt{2n_1}}\,\Phi, \text{ and similarly}\\
\psi &= \frac{1}{\sqrt{2n_2}}\,\Psi,\\
\chi &= \frac{1}{\sqrt{2n_3}}\,X;
\end{aligned}
\tag{42}
$$

only two of the signs of these formations are arbitrary for $2\,\Phi\,\Psi\,X = T_\Gamma$.

Substituting these values for $\varphi\ \psi\ \chi$ we obtain

$$
\begin{aligned}
\psi \mp \chi &= \frac{1}{\sqrt{2\,n_2\,n_3}}\left(\sqrt{n_3}\,\Psi \mp \sqrt{n_2}\,X\right),\\
l_3\,\psi \mp l_2\,\chi &= \frac{\sqrt{n_1}}{\Omega^{\frac{3}{2}}\,i\sqrt{3}}\left[\sqrt{n_3}\,(n_1 - n_2)\,\Psi \mp \sqrt{n_2}\,(n_3 - n_1)\,X\right],\\
\overline{l}_3\,\psi \mp \overline{l}_2\,\chi &= \frac{\sqrt{n_1}}{\Omega^{\frac{3}{2}}}\left[\sqrt{n_3^3}\,\Psi \mp \sqrt{n_2^3}\,X\right],
\end{aligned}
$$

$$\text{and } \Gamma = \frac{2}{n_2 - n_3}\,(X^2 - \Psi^2);$$

and substituting these values in the result of the last paragraph we have the following general theorem: —

---

[1]) Bolza l. c. (5), (15) and (85). [2]) Ibid (72).

Theorem. If Γ be any biquadratic quantic and Φ, Ψ, and X be the three quadratic factors of T_Γ, n_1, n_2, and n_3 the roots of the resolvent cubic of Γ, and g an arbitrary quadratic, then the most general reducible hyperelliptic integral has the form

$$\frac{3}{(n_2-n_3)^{1/2}}\int\frac{(\sqrt{n_3}\,\Psi-\sqrt{n_2}\,X)\,(xdx)}{\sqrt{[\sqrt{n_3}(n_1-n_2)\,\Psi+\sqrt{n_2}(n_3-n_1)\,X]\,[(\Gamma g)_2\,\Gamma-\frac{1}{2}H_\Gamma\,g]}} \tag{43}$$

and reduces to the elliptic integral

$$\int\frac{(ydy)}{\sqrt{[\sqrt{n_3^3}\,\Psi_{(y)}+\sqrt{n_2^3}\,X_{(y)}]\,g_{(y)}}}$$

by the transformation

$$y_1=\tfrac{1}{4}\frac{\partial\Gamma}{\partial x_2},$$
$$y_2=-\tfrac{1}{4}\frac{\partial\Gamma}{\partial x_1}.$$

This form of the reducible integral is now completely independent of the cubic involution.

§ 7. A third general form of the reducible integral independent of the Involution.

Finally we will express the integral in terms of new functions, ξ, η, and ζ, defined in the following manner:—

$$\left\{\begin{aligned}\Phi&=\sqrt{\tfrac{1}{2}A^1_{00}}\;\xi;\\ \Psi&=\sqrt{\tfrac{1}{2}A^1_{11}}\;\eta;\\ X&=\sqrt{\tfrac{1}{2}A^1_{22}}\;\zeta;\end{aligned}\right. \tag{44}$$

where $A^1_{00}=[\Phi\Phi]_2$; $A^1_{11}=[\Psi\Psi]_2$; $A^1_{22}=[XX]_2$;
whence it follows at once that

$$\begin{aligned}A_{\xi\xi}&\equiv[\xi\xi]_2=2,\\ A_{\eta\eta}&\equiv[\eta\eta]_2=2,\\ A_{\zeta\zeta}&\equiv[\zeta\zeta]_2=2;\end{aligned} \tag{46}$$

also $A^1_{11}=\frac{1}{2}(n_1-n_2)(n_2-n_3)=\dfrac{\Omega}{2}(\overline{l_2}-\overline{l_3})(\overline{l_1}-\overline{l_2})=-\frac{3}{2}\,l_1\,l_3\,\Omega$;

and $A^1_{22}=-\frac{3}{2}\,l_1\,l_2\,\Omega$.

∴ We have: —

$$\begin{aligned} \Phi &= \frac{i\sqrt{3}}{2}\,\Omega^{1/2}\sqrt{l_2 l_3}\;\xi; \\ \Psi &= \frac{i\sqrt{3}}{2}\,\Omega^{1/2}\sqrt{l_3 l_1}\;\eta; \\ X &= \frac{i\sqrt{3}}{2}\,\Omega^{1/2}\sqrt{l_1 l_2}\;\zeta; \end{aligned} \tag{46}$$

since the quadratics Φ, Ψ, and X are mutually harmonic it follows that ξ, η, ζ, are also harmonic; we know also the discriminant of each is equal to 2. (45).

We have to express $[g\Gamma]_2\,\Gamma - \frac{1}{2}\,H_\Gamma\,g$ in terms of ξ, η, ζ.

Since $\Gamma = K\,[l_3\eta^2 - l_2\zeta^2]$, where $K = \frac{\sqrt{3}}{2i}\,\Omega^{1/2}$,

$$\begin{aligned} [g\Gamma]_2 &= K\left[l_3(g\eta^2)_2 - l_2(g\zeta^2)_2\right] \\ &= K\left[(l_3 B\eta - l_2 C\zeta) - \tfrac{2}{3}\,(l_3 - l_2)\,g\right] \end{aligned}$$

where $B = (\eta g)_2$ and $C = (\zeta g)_2$.

$$H_\Gamma = \tfrac{2}{3}\,[\eta^2.\,l_3\,(l_2 - l_1) + \zeta^2.\,l_2\,(l_3 - l_1)]\;K^2;$$

$$\therefore\;(g\Gamma)_2\,\Gamma - \tfrac{1}{2}\,H_\Gamma\,g \equiv K^2\,[(l_3 B\eta - l_2 C\zeta)(l_3\eta^2 - l_2\zeta^2) - (l_3^2\eta^2 + l_2^2\zeta^2)g] \tag{47}$$

where $K^2 = -\frac{3}{4}\,\Omega = -3\,\bar{l}_1^2\,\bar{l}_2^2\,\bar{l}_3^2$ § 5. (23)

We find also

$$\sqrt{n_3}\;\Psi - \sqrt{n_2}\;X = \frac{i\sqrt{3}}{2}\sqrt{l_1}\;\Omega^{1/4}\left[\sqrt{l_3\bar{l}_3}\,\eta - \sqrt{l_2\bar{l}_2}\,\zeta\right];$$

$$\bar{n}_3\sqrt{n_3}\;\Psi + \bar{n}_2\sqrt{n_2}\;X = \frac{i\sqrt{3}}{2}\sqrt{l_1}\;\Omega^{3/4}\,\bar{\Omega}^{1/2}\left[\sqrt{l_3^3\bar{l}_3}\;\eta + \sqrt{l_2^3\bar{l}_2}\,\zeta\right];$$

$$\sqrt{n_3^3}\;\Psi + \sqrt{n_2^3}\;X = \frac{i\sqrt{3}}{2}\sqrt{l_1}\;\Omega^{5/4}\left[\sqrt{\bar{l}_3^3 l_3}\,\eta + \sqrt{\bar{l}_2^3 l_2}\,\zeta\right];$$

and $\Gamma = \frac{\sqrt{3}}{i}\;\bar{l}_1\,\bar{l}_2\,\bar{l}_3\,(l_3\eta^2 - l_2\zeta^2)$;

substitute these values in § 6. (43) and we obtain the theorem: —

Theorem. If ξ, η, and ζ be three quadratic quantics mutually harmonic and so normalized that the discriminant of each is equal to 2, and if also $l_1, l_2, l_3, \bar{l}_1, \bar{l}_2, \bar{l}_3$ be any quantities which satisfy the four relations: —

$$l_1 + l_2 + l_3 = 0;\; i\sqrt{3}\,\bar{l}_1 = l_2 - l_3;\; i\sqrt{3}\,\bar{l}_2 = l_3 - l_1;$$
$$i\sqrt{3}\,\bar{l}_3 = l_1 - l_2:$$

and if g be any arbitrary quadratic, then the most general reducible hyperelliptic integral has the form

$$(48) \quad i\sqrt{3}\int \frac{[\sqrt{\bar{l}_3 l_3}\,\eta - \sqrt{\bar{l}_2 l_2}\,\zeta]\,(xdx)}{\sqrt{(\sqrt{l_3^3 \bar{l}_3}\,\eta + \sqrt{l_2^3 \bar{l}_2}\,\zeta)\,[[l_3 B\eta - l_2 C\zeta]\,[l_3\eta^2 - l_2\zeta^2] - g\,[l_3^2\eta^2 + l_2^2\zeta^2]]}}$$

(where B and C are constants equal to $(\eta g)_2$ and $(\zeta g)_2$ respectively) and reduces to the elliptic integral

$$\int \frac{(ydy)}{\sqrt{(\sqrt{\bar{l}_3^3\, l_3}\,\eta(y) + \sqrt{\bar{l}_2^3\, l_2}\,\zeta(y))\;g(y)}}$$

by the cubic transformation

$$l_3\,\eta'^2_x\,\eta_x\,\eta_y - l_2\,\zeta'^2_x\,\zeta_x\,\zeta_y = 0$$

$$\text{where } \eta^2_x = \eta'^2_x = \eta$$

$$\zeta^2_x = \zeta'^2_x = \zeta.$$

This form is again completely independent of the involution.

Part II.

In this second part, we propose to determine all those singular cases of simultaneous reduction, where the octavic, upon which the hyperelliptic integral depends, is the same for two integrals, one of which is reducible by a transformation of the second degree and the other, by one of the third.

§ I, General case of reduction of the second degree.

For this purpose, we first determine the general form of a hyperelliptic integral, of the first kind, and of genus three, which is reducible by a rational transformation of the second degree.

If we use again the method of Jacobi[1]), just as in Part I, § 1, we deduce the following theorem.

Theorem. In order that the hyperelliptic integral

$$\int \frac{q(x_1\, x_2)\,(xdx)}{\sqrt{R(x_1\, x_2)}}$$

where $q\,(x_1\, x_2)$ is a quadratic and $R\,(x_1\, x_2)$ an octavic quantic, shall be reducible to the elliptic integral

$$M \int \frac{(ydy)}{\sqrt{(y_1 - \lambda_1\, y_2)\,(y_1 - \lambda_2\, y_2)\,(y_1 - \lambda_3\, y_2)\,(y_1 - \lambda_4\, y_2)}} \tag{49}$$

M being a constant, by the transformation

$$y_1 = U$$
$$y_2 = V,$$

where U and V are quadratic quantics in $(x_1\, x_2)$, it is necessary and sufficient that $R\,(x_1 x_2)$ shall be decomposable into four quadratic factors belonging to the same quadratic involution, and that the quantic $q\,(x_1 x_2)$ shall have as roots the double points of that involution.

By a parametric representation of the roots of $R\,(x_1 x_2)$ as points on a conic section, we have a geometrical statement of this condition for reduction of the second degree: —

---

[1]) Hutchinson, l. c. § (1).

The points of $R\ (x_1 x_2)$ must be so situated, that we can pair them in such a way that the four joins shall pass through a common point, and that the tangents to the conic, from this point, shall touch in points, which represent the roots of $q\,(x_1 x_2)$.

§ 2. Singular cases of reduction of the second degree.

From the conditions for reducibility determined in § 1 of this part, we conclude that in general for a given $R\ (x_1 x_2)$, which fulfills the required conditions of § 1, there exists but one reducible integral. It may happen, however, that $R\ (x_1 x_2)$ is decomposable in the required manner in more than one way, then there would exist more than one hyperelliptic integral of the first kind, with the same octavic $R\ (x_1 x_2)$, which are reducible to elliptic integrals.

These singular cases can be determined by exactly the same method which J. I. Hutchinson has used in his dissertation[1]), where he determines the singular case of hyperelliptics of the first kind and genus $p = 2$, reducible by a transformation of the second degree.

Accordingly we first determine all octavics, which are invariant under finite linear substitution groups.

For each of the octavics so obtained, pick out all the transformations of period 2, which interchange the eight roots in pairs, and for each such transformation find the double or stationary points. Having all these transformations we can obtain all reducible integrals.

The numerator of the hyperelliptic in each case, is that quadratic, whose roots are the double points of the transformation.

The radical is, of course, always $\sqrt{R\,(x_1 x_2)}$ and we may use for the transformation which is to reduce the hyperelliptic integral

$$\begin{aligned} y_1 &= (xd_1)^2 \\ y_2 &= (xd_2)^2 \end{aligned}$$

where $d_1\, d_2$ are the double points of the transformation.

The following table contains an enumeration of all such octavics, the groups under which each is invariant, the stationary points and also the number of distinct reducible hyperelliptic integrals, which can be obtained for each octavic.

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[1]) On the Reduction of Hyperelliptic Functions ($p = 2$) etc. § 2. Chicago 1897.

R	Group	Equation of double or stationary points	No. of reducible integrals
I(a) $x_1^8 + 4\,ax_1^6x_2^2 + 6\,bx_1^4x_2^4 + 4\,cx_1^2x_2^6 + dx_2^8$	Cyclic $n = 2$	$x_1x_2 = 0$	1
I(b) $x_1x_2\,(x_1^6 + ax_1^4x_2^2 + bx_1^2x_2^4 + cx_2^6)$	Cyclic $n = 2$		none
II $x_1\,(x_1^7 + x_2^7)$	Cyclic $n = 7$		none
III(a) $x_1^8 + ax_1^6x_2^2 + bx_1^4x_2^4 + ax_1^2x_2^6 + x_2^8$	Dihedron $n = 2$	$x_1x_2 = 0$; $x_1^2 = x_2^2$; $x_1^2 = -\,x_2^2$	3
III(b) $x_1x_2\,(x_1^2 + x_2^2)\,(x_1^4 + ax_1^2x_2^2 + x_2^4)$	Dihedron $n = 2$	$x_1^2 = x_2^2$	1
IV $x_1x_2\,(x_1^6 + ax_1^3x_2^3 + x_2^6)$	Dihedron $n = 3$	$x_1^2 = e^{\frac{2n}{3}\pi i}x_2^2$; $(n = 0, 1, 2.)$	3
V $x_1^8 + ax_1^4x_2^4 + x_2^8$	Dihedron $n = 4$	$x_1x_2 = 0$; $x_1^2 = e^{\frac{n\pi i}{2}}x_2^2$ $(n = 0,1,2,3)$	5
VI $x_1x_2\,(x_1^6 + x_2^6)$	Dihedron $n = 6$	$x_1^2 = e^{\frac{2n+1}{3}\pi i}\,x_2^2$ $(n = 0, 1, 2)$	3
VII $x_1^8 - x_2^8$	Dihedron $n = 8$	$x_1^2 = e^{\frac{2n+1}{4}\pi i}\,x_2^2$ $(n = 0,1,2,3)$ $x_1x_2 = 0$	5
VIII $x_1^8 + 14\,x_1^4x_2^4 + x_2^8$	Octahedron	$x_1x_2 = 0$; $x_1^2 = x_2^2$; $x_1^2 = -\,x_2^2$ $x_1^2 = ix_2^2$; $x_1^2 = -\,ix_2^2$ $x_1^2 - 2ix_1x_2 + x_2^2 = 0$; $x_1^2 - 2ix_1x_2 - x_2^2 = 0$ $x_1^2 + 2ix_1x_2 + x_2^2 = 0$; $x_1^2 + 2ix_1x_2 - x_2^2 = 0$	9

§ 3. Cases of simultaneous reduction of the second and third degree.

After these preliminaries we turn to the main problem of this section, namely the determination of all cases where there exist two hyperelliptic integrals of the first kind and genus 3, which have the same $R\ (x_1\, x_2)$, and where one is reducible by a transformation of the second degree, while the other is reducible by one of the third.

We will consider this from a geometrical standpoint.

By means of the parametric representation of a point on a conic section C_2, we have the following geometric interpretation of a cubic involution due to Weyr.

The three points of a triple of the involution define a triangle, which is inscribed in the conic C_2, and circumscribed about a conic I_2, called the involution conic.

The points of intersection of C_2 and I_2, are the branch points of the involution, and the points, where the common tangents to C_2 and I_2 touch C_2, are the double points.

If we choose for the triangle of reference the self conjugate triangle of I_2, the conics assume the equations[1])

$$\begin{aligned} &(C_2) && x^2 + y^2 + z^2 = 0 \\ &(I_2) && \frac{x^2}{l_1^2} + \frac{y^2}{l_2^2} + \frac{z^2}{l_3^2} = 0 \end{aligned} \tag{50}$$

The coordinates of the double points are

$$\begin{aligned} \varrho x &= \sqrt{l_1 \bar{l}_1}\ ; \\ \varrho y &= \pm \sqrt{l_2 \bar{l}_2}\ ; \\ \varrho z &= \pm \sqrt{l_3 \bar{l}_3}\ ; \end{aligned} \tag{51}$$

and those of the branch points are

$$\begin{aligned} \varrho x &= l_1 \sqrt{l_1 \bar{l}_1}\ ; \\ \varrho y &= \pm\, l_2 \sqrt{l_2 \bar{l}_2}\ ; \\ \varrho z &= \pm\, l_3 \sqrt{l_3 \bar{l}_3}\ . \end{aligned} \tag{52}$$

If therefore the octavic can be broken up in the two ways spoken of above, the following conditions will be imposed upon the positions of the eight points of the octavic on the conic C_2.

Since the cubic reduction is to take place, three of the points of R, — A, B, and C — must define a triangle, inscribed in C_2, and circumscribed about the conic I_2, a second triple must determine another such triangle $A' B' C'$, and the remaining two points of R, must be two of the points of intersection of the conics C_2 and I_2, say D and E. But since the reduction of the second degree is also to be possible, the eight points of R must be so situated that we can pair them in such a way that the joins of the four pairs shall pass through a common point, O.

---

[1]) Bolza, l. c. pg. 90.

Since A, B, and C are equivalent to one another, and likewise A', B' and C', and also D and E, we obtain in all, the following five essentially different ways of pairing the eight points: —

$$\begin{array}{rllll} \text{I.} & AA' ; & BB' ; & CC' ; & DE. \\ \text{II.} & AB ; & A'B' ; & CC' ; & DE. \\ \text{III.} & AB ; & A'B' ; & CD ; & C'E. \\ \text{IV.} & AA' ; & B'C' ; & BD ; & CE. \\ \text{V.} & AA' ; & BB' ; & CD ; & C'E. \end{array} \tag{53}$$

§ 4. Case I. AA'; BB'; CC'; DE.

Since these pairs of points determine lines which all pass through a common point O: we can, by taking as the fundamental points of the binary system of coordinates on C_2 the points of tangency on C_2 of the two tangents from O, obtain the following parametric representation of these eight points upon C_2: —

$$\begin{array}{llll} A \frown k_1 : k_2 & & A' \frown -k_1 : k_2 \\ B \frown l_1 : l_2 & & B' \frown -l_1 : l_2 \\ C \frown m_1 : m_2 & & C' \frown -m_1 : m_2 \\ D \frown n_1 : n_2 & & E \frown -n_1 : n_2 . \end{array} \tag{54}$$

Now $A\,B\,C$ and $A'\,B'\,C'$ are triples of a cubic involution and we find the Jacobian, or double point equation: —

$$\vartheta = -\tfrac{2}{3}\,[a_0\,a_1\,x_1^4 + (a_0\,a_3 + 3\,a_1\,a_2)\,x_1^2\,x_2^2 + a_2\,a_3\,x_2^4] \tag{55}$$

$$\text{where}\quad \begin{aligned} a_0 &= k_2\,l_2\,m_2 \\ 3\,a_1 &= \Sigma\,k_1\,l_2\,m_2 \\ 3\,a_2 &= \Sigma\,k_1\,l_1\,m_2 \\ a_3 &= k_1\,l_1\,m_1 \end{aligned}$$

and the branch equation is: —

$$6\,a_0\,a_1\,(a_0\,a_3 - a_1\,a_2)\,x_1^4 + (5\,a_0^2\,a_3^2 - 30\,a_0\,a_1\,a_2\,a_3 - 27\,a_1^2\,a_2^2)\,x_1^2\,x_2^2 + 6\,a_2\,a_3\,(a_0\,a_3 - a_1\,a_2)\,x_2^4 = 0. \tag{56}$$

∴ We see from the form of (56) that if p is a branch point, then $-p$ is also. Hence, —

When in a cubic involution two triples are such that we can form, by taking one from each, three couples in quadratic involution, then the branch points of that cubic involution form two couples in the quadratic involution.

From this it follows, that O is the point of intersection of DE with the join of the remaining two points of intersection of C_2 and I_2, and hence according to the theory of two conics, the point O is one of the vertices of the self-conjugate triangle, — that is, the lines AA', BB' and CC' must pass through this point, whence,

Theorem: — If the points of any two triples of a cubic involution are in perspective, the centre of perspectivity is a vertex of the self-conjugate triangle of the conic C_2 and the involution conic I_2.

Vice versa, —

Let O be one of the vertices of the self-conjugate triangle, join it to the three points A B C of any triple of the involution; the lines will cut C_2 again in L, M, N: prove that the points L, M, N form a triple also.

The points where the sides of the self-conjugate triangle cut the conic C_2, are given by the roots of the functions φ', ψ', χ', 1) which are the factors of T_B, where B is a biquadratic whose roots are the points of intersection of the conics C_2 and I_2 i. e. is the branch equation.

These functions φ' ψ' and χ' can be normalized so as to take the form

$$\begin{aligned} \varphi' &= C_1\,(x_1 x_2) \\ \psi' &= C_2\,(x_1^2 + x_2^2) \\ \chi' &= C_3\,(x_1^2 - x_2^2) \quad \text{where the } C\text{'s are constants.} \end{aligned} \tag{57}$$

Let us take for O the vertex of the self-conjugate triangle opposite side φ'; then the lines drawn through the point O cut the conic C_2 in point pairs of the involution, $\lambda_1 x_1^2 + \lambda_2 x_2^2 = 0$.

The transformation which interchanges the points of a pair is

$$x_1' = -x_1 \qquad x_2' = x_2.$$

and hence, if the parameters for the points $A, B, C,$ be $q, r, s,$ joining these to O, the parameters of the corresponding points M, N, L must be $-q, -r, -s$.

And we can prove that if q, r, s is a triple, then is also $-q, -r, -s$, a triple, of the cubic involution.

For the quadratics $\varphi' \psi' \chi'$ which we derived from the branch form

$$3\,H + J\vartheta$$

are found to be proportional to the similar factors, φ, ψ, χ of T_ϑ; for T_ϑ is proportional to $T_{3H + J\vartheta}$.

And since we have an expression for Γ in terms of $\varphi, \psi,$ and χ (24), which by using the normal form of φ, ψ, χ (57), assumes a form which involves only even powers of the variables; and hence, since all triples of our involution are given as roots of the equation

$$\lambda_1 \frac{\partial \Gamma}{\partial x_1} + \lambda_2 \frac{\partial \Gamma}{\partial x_2} = 0$$

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1) W. S. Burnside. — On the Invariants and Covariants of a Binary Quartic etc. — Quarterly-Journal Vol. 10. pag. 211.

F. Meyer, Apolarität. § 57.

if $q\,r\,s$ forms a triple, then, by merely changing the sign of $\lambda_1 : \lambda_2$, $-q, -r, -s$ must also form a triple.

Theorem. If we project a triple of the involution from a vertex of the self-conjugate triangle upon the normal conic, we obtain another triple.

Thus we have a perfectly definite position O, and so, a solution of our problem.

Being given ϑ and the sign of J which are necessary for the construction of the conics C_2 and I_2, we may take for O any vertex of the self-conjugate triangle.

Then choosing arbitrarily any point A on the conic C_2, the two tangents to I_2 from A will determine B and C, and projecting the points A, B, and C from O we obtain another triple, A', B', and C'. This was the required arrangement.

One point A is chosen arbitrarily, and for O we have a choice of three points, but after that all is completely fixed.

In each of the following cases there is a finite number of solutions, but here since A is arbitrarily chosen we have a single infinitude.

By comparing this with the general case of reducible integrals given in § 7, Part I, let us see where the condition is imposed.

In the general case, the two triples of the octavic are quite independent, being given by

$$\Gamma_x^3 \; \Gamma_\lambda \text{ and } \Gamma'^3_x \; \Gamma'_\mu$$

where λ and μ are the roots an arbitrarily chosen quadratic $g(x)$.

In this special case the Γ has the normal form

$$\Gamma = -i\sqrt{3}\;\overline{l_1}\;\overline{l_2}\;\overline{l_3}\;\left[(x_1^4 + x_2^4)\;l_2 - 2\,i\sqrt{3}\;\overline{l_2}\,x_1^2\,x_2^2\right],$$

and in order to give the proper arrangement of points, μ must $= -\lambda$, and $\therefore$ $g(x)$ has the form $x_1^2 - \lambda^2 x_2^2$; but since in the above normal form $\varphi' = c_1\,2\,x_1 x_2$, $g(x)$ is apolar to φ, and since φ' is proportional to Φ, § 6, P. I and Φ is proportional to ξ, § 7, P. I., the condition for such a reduction is that $g(x)$ shall be apolar to ξ.

Whence we have the theorem:

In order that the most general reducible integral § (7) shall specialize to the present case of simultaneous reduction by a cubic and by a quadratic transformation, it is necessary and sufficient that the quadratic $g(x)$ shall be chosen apolar to the quadratic ξ.

§ 5. Case II. AB, $A'B'$ CC' and DE.

Set us take O $(x_1 y_1 z_1)$ arbitrarily in the plane. Draw the two tangents to I_2 from O; one of them cuts C_2 in the points A and B, the other in the points $A' B'$. Let the points of tangency of these lines be K and K'.

According to (50) KK' has the equation

$$\frac{xx'}{l_1^2} + \frac{yy'}{l_2^2} + \frac{zz'}{l_3^2} = 0, \tag{59}$$

whence we have the coordinates of K and K'

$$\begin{aligned} \varrho\overline{x} &= -\frac{x_1 z_1}{l_3^2} \pm i\frac{y_1 l_1}{l_2 l_3}\sqrt{I_2(x_1 y_1 z_1)} \\ \varrho\overline{y} &= -\frac{y_1 z_1}{l_3^2} \mp i\frac{x_1 l_2}{l_1 l_3}\sqrt{I_2(x_1 y_1 z_1)} \\ \varrho\overline{z} &= \frac{x_1^2}{l_1^2} + \frac{y_1^2}{l_2^2} \end{aligned} \tag{60}$$

$$\text{where } I_2(x_1 y_1 z_1) \equiv \frac{x_1^2}{l_1^2} + \frac{y_1^2}{l_2^2} + \frac{z_1^2}{l_3^2};$$

we have however the following relation between the coordinates $\overline{x}\ \overline{y}\ \overline{z}$ of K and those of C, xyz;

$$\begin{aligned} \overline{x} &= l_1 x \\ \overline{y} &= l_2 y \\ \overline{z} &= l_3 z \end{aligned} \tag{61}$$

(Bolza. l. c. 60(a).

Whence we obtain for C and C'

$$\left\{ \begin{aligned} \varrho x &= -\frac{x_1 z_1}{l_1 l_3} \pm \frac{yi}{l_2}\sqrt{I_2(x_1 y_1 z_1)} \\ \varrho y &= -\frac{y_1 z_1}{l_2 l_3} \mp \frac{xi}{l_1}\sqrt{I_2(x_1 y_1 z_1)} \\ \varrho z &= \frac{x_1^2}{l_1^2} + \frac{y_1^2}{l_2^2} \end{aligned} \right. \tag{62}$$

∴ the equation of CC' is

$$\frac{xx_1}{l_1} + \frac{yy_1}{l_2} + \frac{zz_1}{l_3} = 0, \tag{63}$$

or it is the polar of the point O $(x_1 y_1 z_1)$ with respect to the conic, B_2

$$\frac{x^2}{l_1} + \frac{y^2}{l_2} + \frac{z^2}{l_3} = 0 \quad (B_2). \tag{64}$$

But to fulfill the conditions of our problem CC' must pass through O, ∴ O must lie on the conic B_2.

But O must also lie on the line DE.

Therefore the points of intersection of B_2 and DE are admissible positions for O.

Taking for D and E the points,

$$\begin{aligned} &(D)\quad \varrho x = l_1\sqrt{l_1\overline{l}_1}; &\quad &(E)\quad \varrho x = l_1\sqrt{l_1\overline{l}_1}; \\ &\quad \varrho y = l_2\sqrt{l_2\overline{l}_2}; & &\quad \varrho y = l_2\sqrt{l_2\overline{l}_2}; \\ &\quad \varrho z = l_3\sqrt{l_3\overline{l}_3}; & &\quad \varrho z = -l_3\sqrt{l_3\overline{l}_3}; \end{aligned} \tag{65}$$

the points of intersection are: —

$$\begin{aligned} \varrho x &= l_1 \sqrt{l_1 \overline{l}_1} \quad ; \\ \varrho y &= l_2 \sqrt{l_2 \overline{l}_2} \quad ; \\ \varrho z &= \pm\, i \sqrt{l_3 \overline{l}_3 \, (l_1 l_2 + l_3^2)} \quad ; \end{aligned} \tag{66}$$

of course we may take any two points of intersection of C_2 and I_2 for our cutting line.

§ 6. Properties of the conic B_2.

This conic B_2 has some peculiar properties, and is closely connected with the involution.

I. The conic B_2 passes through the double points of the involution on C_2.

The equation of B_2 is

$$\frac{x^2}{l_1} + \frac{y^2}{l_2} + \frac{z^2}{l_3} = 0 \tag{64}$$

which is satisfied by the four double points (51).

II. The conic B_2 is apolar[1]) to I_2.

The equation of B_2 in point coordinates is

$$\frac{x^2}{l_1} + \frac{y^2}{l_2} + \frac{z^2}{l_3} = 0.$$

The equation of I_2 in tangential coordinates is

$$l_1^2 u^2 + l_2^2 v^2 + l_3^2 w^2 = 0$$

and since $\sum\limits_i l_i^2 \dfrac{1}{l_i} = 0.$

$\therefore$ B_2 is apolar to I_2 . B_2 is said to support the conic I_2, and I_2 to rest on B_2.

This condition, together with I, is sufficient to fix B_2 completely.

B_2 must have the form, $C_2 + \lambda D_2$, where D_2 is some fixed conic through the four double points of the involution.

These two conics C_2 and D_2 being in point coordinates the condition for a polarity of B_2 and I_2,

$$[C_2 + \lambda D_2 \,.\, I_2] = 0$$

is linear in λ, and therefore sufficient to determine λ uniquely.

The conic B_2 is also apolar to C_2.

By taking B_2 as a line conic, and C_2 as a point conic, we find that C_2 supports B_2, and B_2 rests on C_2; but in this case the condition for apolarity will not yield a linear equation to determine the constant λ.

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[1]) Meyer, Apolarität and Rationale Curven pag. 84.

III. The geometrical meaning of apolarity of two conics is that a simply infinite system of triangles, circumscribed about the line conic, are self-conjugate with regard to the point conic.

Here this means that there is a single infinitude of triangles, circumscribed about I_2, self-conjugate with regard to the conic B_2.

Indeed, we will find these triangles are the triangles of involution, viz. those inscribed in C_2 and circumscribed about I_2.

For if $x_1 y_1 z_1$ be any point on C_2, the opposite side of the triangle has the equation

$$\frac{xx_1}{l_1} + \frac{yy_1}{l_2} + \frac{zz_1}{l_3} = 0 .$$

Since this is the equation of the tangent to I_2 at the point $l_1 x_1, l_2 y_1, l_3 z_1$, which is the corresponding point on I_2 to $x_1 y_1 z_1$ on C_2 (61), being also the polar of the point $x_1 y_1 z_1$ with regard to the conic B_2, and since exactly the same reasoning holds for the other two vertices, our triangle, inscribed in C_2 and circumscribed about I_2, is self-conjugate with regard to the conic B_2.

From this theorem it follows at once that the intersection O of two tangents to I_2 cutting C_2 in AB and $A'B'$ respectively, is the pole of the line CC', which was the theorem we proved above by directly solving.

Supposing the cubic involution given, and therefore, the conics I_2, C_2, and B_2, then we have the following general theorem concerning any two triples of the involution: —

Draw from any point O the two tangents to the conic I_2; let them cut the conic C_2 in the points AB and $A'B'$ respectively, then the line joining the points C and C' completing the triples of which AB and $A'B'$ are point-pairs, is the polar of O with respect to the ccnic B_2.

IV. The points in which the common tangents to B_2 and C_2 touch C_2 are the four points, $\overline{\Gamma} = 0$.

The common tangents are found by solving

$$l_1 u^2 + l_2 v^2 + l_3 w^2 = 0$$
$$u^2 + v^2 + w^2 = 0;$$

$$\therefore \varrho u = \pm \overline{l_1}^{1/2}$$
$$\varrho v = \pm \overline{l_2}^{1/2}$$

$\varrho w = \pm \overline{l_3}^{1/2}$; but the tangent to C_2 in the point $x_1 : x_2$ has the coordinates ¹) $\varrho_1 u = \sqrt{m_2 - m_3}\,\varphi = \sigma \sqrt{l_1 \overline{l}_1}\,\varphi$

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¹) Bolza l. c. (61).

$$\text{or}\quad \varrho u = \pm\sqrt{l_1 \bar{l}_1}\,\varphi$$
$$\varrho v = \pm\sqrt{l_2 \bar{l}_2}\,\psi$$
$$\varrho w = \pm\sqrt{l_3 \bar{l}_3}\,\chi$$

∴ for the common tangents $\varrho\varphi = \sqrt{\frac{1}{l_1}}$, $\varrho\psi = \sqrt{\frac{1}{l_2}}$, $\varrho\chi = \sqrt{\frac{1}{l_3}}$.

Now $\bar{\Gamma} = \frac{4i}{\sqrt{3}\,l_1}\left[l_2\psi^2 - l_3\chi^2\right]$ and therefore has these values for roots.

The tangents are

$$\bar{l}_1^{1/2} x \pm \bar{l}_2^{1/2} y \pm \bar{l}_3^{1/2} z = 0.$$

These touch B_2 also, and indeed at the points of intersection of B_2 and I_2,

$$\text{i. c.}\quad \varrho x = \pm l_1\sqrt{\bar{l}_1}.$$
$$\varrho y = \pm l_2\sqrt{\bar{l}_2}.$$
$$\varrho z = \pm l_3\sqrt{\bar{l}_3}.$$

§ 7. Case III. *AB, A′B′, CD, C′E.*

Take arbitrarily the point $S\ (x_1 y_1 z_1)$ in the plane.

Join SE, and SD, and produce these lines to cut the conic C_2 again in C' and C respectively.

Then AB and $A'B'$ are completely defined; indeed by § 3. III we know AB and $A'B'$ are the polars of C and C' with respect to the conic B_2.

Let AB and $A'B'$ intersect in, O then in order that the conditions of our problem be fulfilled O, must coincide with S.

C is the pole of AB with respect to B_2, § (3) III.
C' is the pole of $A'B'$ with respect to B_2 § (3) III.

∴ O the intersection of AB and $A'B'$ is the pole of CC'.

For brevity let us call the coordinates of C, $(x_2 y_2 z_2)$ and those of C', $(x_3 y_3 z_3)$.

Then O is the pole of the line

$$x\,(y_2 z_3 - y_3 z_2) + y\,(z_2 x_3 - z_3 x_2) + z\,(x_2 y_3 - x_3 y_2) = 0$$

with respect to the conic B_2

$$\frac{x^2}{l_1} + \frac{y^2}{l_2} + \frac{z^2}{l_3} = 0.$$

∴ The coordinates of O are

$$\begin{aligned} \varrho x &= l_1\,(y_2 z_3 - z_2 y_3) \\ \varrho y &= l_2\,(z_2 x_3 - z_3 x_2) \\ \varrho z &= l_3\,(x_2 y_3 - x_3 y_2). \end{aligned} \qquad (67)$$

But we can express the coordinates of C and C' in terms of the coordinates of S: —

$$\tag{68} \left\{ \begin{aligned} \varrho x_2 &= z\,x_1\ (ax_1 + by_1 + cz_1) - a\ C_2\,(x_1\,y_1\,z_1) \\ \varrho y_2 &= z\,y_1\ (ax_1 + by_1 + cz_1) - b\ C_2\,(x_1\,y_1\,z_1) \\ \varrho z_2 &= z\,z_1\ (ax_1 + by_1 + cz_1) - c\ C_2\,(x_1\,y_1\,z_1) \end{aligned} \right.$$

$$\tag{69} \left\{ \begin{aligned} \varrho x_3 &= z\,x_1\ (ax_1 + by_1 - cz_1) - a\ C_2\,(x_1\,y_1\,z_1) \\ \varrho y_3 &= z\,y_1\ (ax_1 + by_1 - cz_1) - b\ C_2\,(x_1\,y_1\,z_1) \\ \varrho z_3 &= z\,z_1\ (ax_1 + by_1 - cz_1) + c\ C_2\,(x_1\,y_1\,z_1) \end{aligned} \right.$$

where $C_2\,(x_1\,y_1\,z_1) = x_1^2 + y_1^2 + z_1^2$:
and the coordinates of D are: —

$$\tag{70} \left\{ \begin{aligned} \varrho x &= l_1\,\sqrt{l_1\,\bar{l}_1} = a \\ \varrho y &= l_2\,\sqrt{l_2\,\bar{l}_2} = b \\ \varrho z &= l_3\,\sqrt{l_3\,\bar{l}_3} = c\ : \end{aligned} \right.$$

and those of E are: —

$$\tag{71} \left\{ \begin{aligned} \varrho x &= l_1\,\sqrt{l_1\,\bar{l}_1} = a \\ \varrho y &= l_2\,\sqrt{l_2\,\bar{l}_2} = b \\ \varrho z &= -l_3\sqrt{l_3\,\bar{l}_3} = -c. \end{aligned} \right.$$

Whence for our point O we have from (67) (68) and (69)

$$\tag{72} \begin{aligned} \varrho x &= l_1\,\left[z\,x_1\ (bx_1 - ay_1) - b\,C_2\ (x_1\,y_1\,z_1)\right] \\ \varrho y &= l_2\,\left[z\,y_1\ (bx_1 - ay_1) + a\,C_2\ (x_1\,y_1\,z_1)\right] \\ \varrho z &= l_3\,\left[z\,z_1\ (bx_1 - ay_1)\right]\ ; \end{aligned}$$

but we saw that, for our case, O must coincide with $S\,(x_1\,y_1\,z_1)$, whence: —

$$\tag{73} \begin{aligned} z\,(l_1 - l_3)\ x_1\ (bx_1 - ay_1) &= l_1\,b\,C_2\ (x_1\,y_1\,z_1), \\ z\,(l_3 - l_2)\ y_1\ (bx_1 - ay_1) &= l_2\,a\,C_2\ (x_1\,y_1\,z_1). \end{aligned}$$

The intersection of these two conics (73), will give us four possible positions for O.

Two of the four we can see at once are the points D and E, and such positions for O give rise to degenerate cases only.

The other two points say O_1 and O_2 are: —

$$\tag{74} \begin{aligned} (O_1)\quad \varrho x &= l_2\,\sqrt{l_1\,\bar{l}_1}, & (O_2)\quad \varrho x &= l_2\,\sqrt{l_1\,\bar{l}_1}, \\ \varrho y &= l_1\,\sqrt{l_2\,\bar{l}_2}\,, & \varrho y &= l_1\,\sqrt{l_2\,\bar{l}_2}\,, \\ \varrho z &= 2\,l_3\,\sqrt{l_3\,\bar{l}_3} & \varrho z &= -\,2\,l_3\,\sqrt{l_3\,\bar{l}_3}, \end{aligned}$$

but even these positions give rise to degenerate integrals only, for the join of O_1 to E determines, as C', the double point corresponding to E, viz.

$$\varrho x = \sqrt{l_1\,\bar{l}_1} \qquad \varrho y = \sqrt{l_2\,\bar{l}_2} \qquad \varrho z = -\,\sqrt{l_3\,\bar{l}_3}\,,$$

while the join of O_2 and D determines, as C, the double point corresponding to D, viz.

$$\varrho x = \sqrt{l_1\,\bar{l}_1} \qquad \varrho y = \sqrt{l_2\,\bar{l}_2} \qquad \varrho z = \sqrt{l_3\,\bar{l}_3}\,.$$

Thus we see the integral in each case is degenerate; the radical is over an expresion only of the fourth degree, since one of our triples in each case has become a branch triple, and indeed, one including a branch point already in R_8.

This gives rise to an interesting geometrical theorem: —

Being given the two conics C_2 and I_2, take any point S arbitrarily in the plane, and join S to any two of the points of intersection of C_2 and I_2, say D and E. These lines cut C_2 again in C' and C, and the points C and C' have AB and $A'B'$ as polars with regard to the conic B_2. Let AB and $A'B'$ cut each other in the point O, then the only positions for S, such that O shall coincide with it, are either the points D and E, or two other points, one on each of the tangents to the conic I_2 at these points D and E.

§ 8. Case IV. AA'; $B'C'$; BD; CE.

If from any arbitrarily chosen point S in the plane we draw a tangent to I_2 cutting C_2 in the points $B'C'$, then the point A', which with $B'C'$ forms a triple of the involution, is completely determined; join A' to S, and call the point where the line cuts C_2, A, then the polar of A with regard to B_2 determines the points B, C, forming a triple with A.

Take any two branch points D and E, join BD and CE, and let them intersect in the point O; then in order that the conditions of our case be fulfilled, the point O must coincide with S.

§ 9. Case V. AA'; BB'; CD; $C'E$.

Take any point S arbitrarily in the plane and join it to two branch points D and E; these lines cut C_2 again in two points C and C'. The polars of C and C' with regard to the conic B_2 determine the points AB and $A'B'$ respectively. Let the lines AB and $A'B'$ intersect in a point O, the pole of CC' with regard to the conic B_2; then in order that the conditions of our case be fulfilled, the point O must coincide with S.

The last two cases lead to results too complicated to be given here.

Part III.

§ I. Involutions containing two cubes.

In the cases considered up to this point, we have always supposed the involution to be perfectly general, but let us now consider the specia case, in which the cubic involution contains two cubes.

As Clebsch has shown[1]) every such involution may be written in the form

$$\lambda_1 f + \lambda_2 Q = 0 \tag{75}$$

where $Q = (f\Delta)$ and $\Delta = (ff)_2$.

Then using the notation of Clebsch we find

$$\left\{\begin{array}{l} \vartheta = (fQ) = -\frac{1}{2}\Delta^2; \\ J = (fQ)_3 = R \\ H_\vartheta = \frac{1}{12} R\Delta^2; \\ \Gamma \equiv 0 \\ 3H + J\vartheta = -\frac{1}{2} R\Delta^2. \end{array}\right. \tag{76}$$

From these equations, applying the method of Part I § 1, wel find, that the most general reducible hyperelliptic integral corresponding to this special form of the cubic involution, has the form

$$\int \frac{\Delta\,(xdx)}{\sqrt{\Delta\,(\lambda_1 f + \lambda_2 Q)\,(\mu_1 f + \mu_2 Q)}}\,. \tag{77}$$

The conditions, necessary and sufficient for such a special form of the involution, are

$$J \neq 0 \text{ and } \Gamma \equiv 0.$$

But we can express the involution (75) in the normal form[2])

$$\lambda'_1 x_1^3 + \lambda'_2 x_2^3 = 0. \tag{78}$$

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[1]) Clebsch, l. c. § 39.

[2]) Ibid.

Using this form for the involution, we obtain the equations:

$$\vartheta = x_1^2 x_2^2 \,,\; 3H + J\vartheta = \tfrac{1}{2} x_1^2 x_2^2$$

and it follows, the hyperelliptic integral

$$\int \frac{x_1 x_2 \,(xdx)}{\sqrt{x_1 x_2 \,(\lambda_1' x_1^3 + \lambda_2' x_2^3)\,(\mu_1' x_1^3 + \mu_2' x_2^3)}} \tag{79}$$

is reducible to the elliptic integral

$$\int \frac{(ydy)}{\sqrt{y_1 y_2 \,(\lambda_1' y_1 + \lambda_2' y_2)\,(\mu_1' y_1 + \mu_2' y_2)}}$$

by the transformation

$$\begin{aligned} y_1 &= x_1^3 \\ y_2 &= x_2^3 \,. \end{aligned}$$

§ 2. Cases of simultaneous reduction of the second and third degree.

Is it possible that the octavic

$$\Delta\,(\lambda_1 f + \lambda_2 Q)\,(\mu_1 f + \mu_2 Q),$$

or in its normal form,

$$x_1 x_2 \,(\lambda_1' x_1^3 + \lambda_2' x_2^3)\,(\mu_1' x_1^3 + \mu_2' x_2^3),$$

can be decomposed in such a way as to yield a quadratic reduction of the hyperelliptic integral?

If this be possible, then the octavic must coincide with some one of the invariant octavics found in the table, Part II, § 2.

This is impossible so long as $\lambda_1' : \lambda_2'$ and $\mu_1' : \mu_2'$ remain arbitrary, but if $\lambda_1' : \lambda_2' = \mu_2' : -\mu_1'$, then the octavic assumes a form invariant under the dihedron group ($n = 3$).

By making use of our table, we find this will yield us three reducible integrals.

If, in particular, $\lambda_1' : \lambda_2' = \mu_2' : -\mu_1' = 1$, then the octavic assumes the form

$$x_1 x_2 \,(x_1^6 - x_2^6)\,, \tag{80}$$

invariant under the dihedron group ($n = 6$).

In the same way, we find for this case the following three integrals and as they are obtained later by a completely different method, it is well to state the reduction in full.

The hyperelliptic integrals

$$\int \frac{(x_1^2 - e^{\frac{2n+1}{3}\pi i} x_2^2)\,(xdx)}{\sqrt{x_1 x_2 \,(x_1^6 - x_2^6)}} \quad \text{for } n = 0,\, 1,\, 2, \tag{81}$$

are reducible by the transformation

$$y_1 = (x_1 + e^{\frac{2n+1}{6}\pi i}\, x_2)^2$$

$$y_2 = (x_1 - e^{\frac{2n+1}{6}\pi i}\, x_2)^2$$

to the elliptic integral

$$2\sqrt{2\, e^{\frac{-2n+1}{12}\pi i}} \int \frac{(ydy)}{\sqrt{(y_1^3 + 15\, y_1^2 y_2 + 15 y_1 y_2^2 + y_2^3)\,(y_1 - y_2)}}$$

In order that the octavic shall belong to the dihedron group $n = 6$, we impose the condition that

$$\lambda_1' : \lambda_2' = \mu_2' : -\mu_1' = 1$$

in the triples

$$\lambda_1' f + \lambda_2' Q \text{ and } \mu_1' f + \mu_2' Q.$$

What does this mean for these two cubics? In the normal form

$$\lambda_1' f + \lambda_2' Q = x_1^3 - x_2^3 \text{ and } \mu_1' f + \mu_2' Q = x_1^3 + x_2^3.$$

$$\therefore\ \mu_1' f + \mu_2' Q = \varrho Q_{\lambda_1' \lambda_2'} = \sigma\left(-\frac{R}{2}\lambda_2' f + \lambda_1' Q\right) \text{ [1]}$$

$$\therefore\ \mu_1' = -\frac{R}{2}\lambda_2'$$

$$\mu_2' = \lambda_1'$$

If now we choose our notation so that

$$f_{\lambda_1' \lambda_2'} = f \text{ and } Q_{\lambda_1' \lambda_2'} = Q$$

we have the following reducible hyperelliptic integrals: —

(82) $$\int \frac{a_x^2\, a_{\alpha_i}\, (xdx)}{\sqrt{\Delta Q f}} \qquad i = 1, 2, 3.$$

where $a_x^3 = f$ and $Q = (x\alpha_1)\,(x\alpha_2)\,(x\alpha_3)$

and the theorem: —

For a given octavic $\Delta f Q$ we have altogether four reducible hyperelliptic integrals, one (79) whose numerator is Δ, reducible by a transformation of the third degree, and three others (82) whose numerators are the first polar of f with regard to the roots of Q, by transformations of the second degree.

§ 3. Integrals treated by the method of periods.

In this last part, we purpose to study the reduction problem for a special hyperelliptic curve

$$y^2 = x\,(1 - x^6)$$

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[1] Clebsch, l. c., pag. 123.

by means of the **Weierstrass-Picard** theorem on the periods of reducible integrals.

We choose, as a system of three linearly independent integrals of the first kind,

$$w_1 = \int \frac{dx}{y}, \quad w_2 = \int \frac{x\,dx}{y}, \quad w_3 = \int \frac{x^2\,dx}{y}, \tag{83}$$

where $y^2 = x\,(1 - x^6)$.

Further we choose the cross cuts, which render the **Riemann** surface simply connected, as in the adjoining diagram; a_i $(i = 0, 1, \ldots .\, 6)$ denote the roots of y^2. Such cuts form a canonical system.

$$\text{Setting } I_\lambda = \int_{a_\lambda}^{a_\lambda - 1} \frac{x^{\alpha - 1}\,dx}{y_1}$$

where y_1 denotes the value of y on the positive side of cut, and

$$P_\mu = \int_{a_6}^{a_\mu} \frac{x^{\alpha - 1}\,dx}{y_1}$$

Then if

$2\,\omega_{\alpha\beta}$ = the modulus of periodicity of w_α at A_β cut,

and $2\,\omega'_{\alpha\beta}$ = the modulus of periodicity of w_α at B_β cut,

we have: (See for instance **Appell** and **Goursat**)

$$\begin{aligned}
\omega'_{\alpha 1} &= I_2 = P_1 - P_2 \\
\omega'_{\alpha 2} &= I_4 = P_3 - P_4 \\
\omega'_{\alpha 3} &= I_6 = P_5 \\
\omega'_{\alpha 1} &= I_1 = P_1 - P_0 \\
\omega'_{\alpha 2} &= I_1 + I_3 = P_1 + P_3 - (P_0 + P_2) \\
\omega'_{\alpha 3} &= I_1 + I_3 + I_5 = P_1 + P_3 - P_5 - (P_0 + P_2 + P_4)
\end{aligned} \tag{84}$$

We find easily $P_\lambda = K_\alpha\,\Theta^\lambda$

$$\text{where } K_\alpha = \frac{\sqrt{\pi}}{6}\,\frac{\Gamma\frac{2\alpha - 1}{12}}{\Gamma\frac{(2\alpha + 2p - 1)}{12}} \tag{85}$$

and $\Theta = j^{2\alpha - 1} = e^{(2\alpha - 1)\frac{i\pi}{6}}$.

We will choose for our integrals to work with

$$W_1 = \frac{w_1}{K_1}, \quad W_2 = \frac{w_2}{K_2}, \quad \text{and } W_3 = \frac{w_3}{K_3}. \tag{86}$$

For W_α $(\alpha = 1, 2, 3)$ we have the table of half periods

$$\begin{aligned}
\omega_{\alpha_1} &= \Theta\,(1-\Theta) & \omega'_{\alpha 1} &= \Theta - 1 \\
\omega_{\alpha_2} &= \Theta^3\,(1-\Theta) & \omega'_{\alpha 2} &= \frac{\Theta^4-1}{\Theta+1} \\
\omega_{\alpha_3} &= \Theta^5 & \omega'_{\alpha 3} &= \frac{\Theta^6-1}{\Theta+1}
\end{aligned}$$

Hence we deduce the table of half periods:

$$\begin{array}{c|cccccc}
 & A_1 & A_2 & A_3 & B_1 & B_2 & B_3 \\
\hline
W_1 & j\,(1-j)\,; & j^3\,(1-j)\,; & j^5\,; & j-1\,; & \dfrac{j^4-1}{j+1}\,; & \dfrac{j^6-1}{j+1} \\
W_2 & j^3\,(1-j)\,; & j^9\,(1-j^3)\,; & j^{15}\,; & j^3-1\,; & \dfrac{j^{12}-1}{j^3+1}\,; & \dfrac{j^{18}-1}{j^3+1} \\
W_3 & j^5\,(1-j^5)\,; & j^{15}\,(1-j^5)\,; & j^{25}\,; & j^5-1\,; & \dfrac{j^{20}-1}{j^5+1}\,; & \dfrac{j^{30}-1}{j^5+1}
\end{array} \tag{87}$$

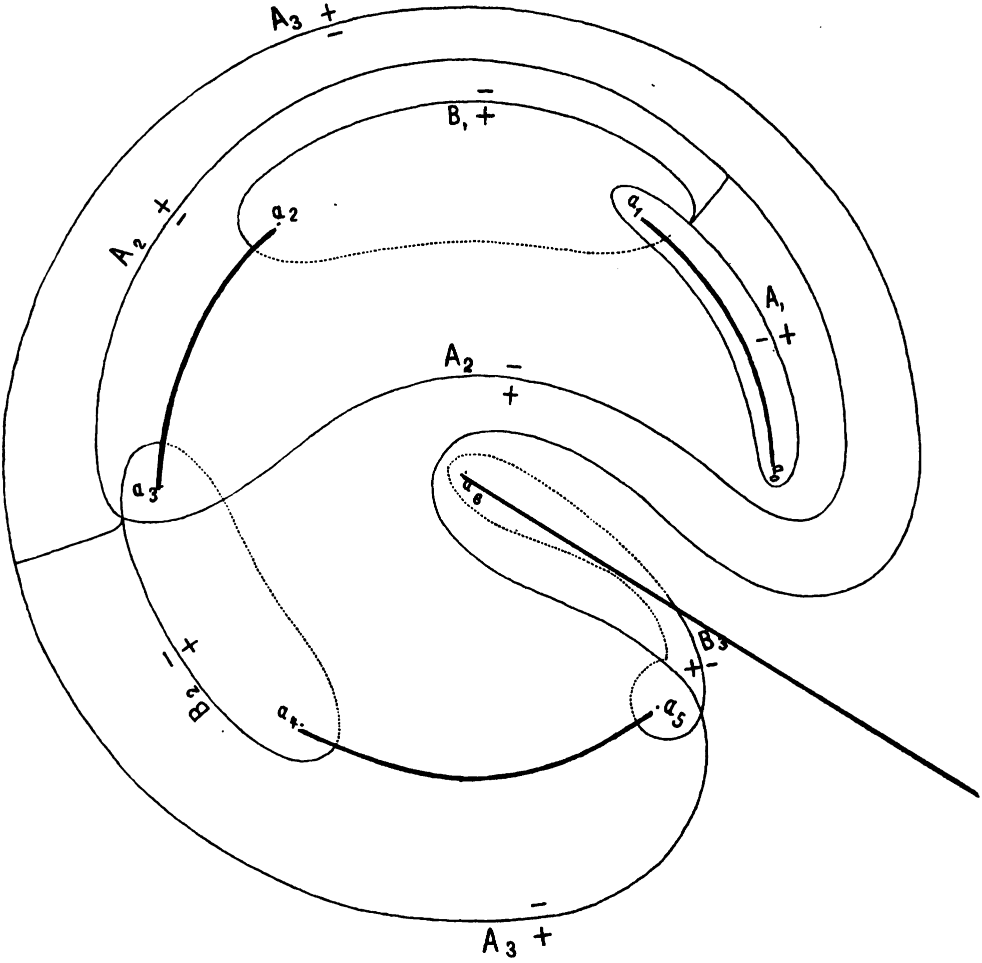
Now denoting by $\overline{\omega}_{\alpha\beta}$ the half period of W_α at the cross cut A_β
and $\overline{\omega}'_{\alpha\beta}$ the half period of W_α at the cross cut B_β
we obtain after a little computation

$$\left\{\begin{array}{ll}
W_1 & \left\{\begin{aligned}
\overline{\omega}_{11} &= \frac{\sqrt{3}-1}{2}\,(1-i) & \overline{\omega}'_{11} &= \frac{\sqrt{3}+i}{2} - 1 \\
\overline{\omega}_{12} &= \tfrac{1}{2} + i\,(1-\frac{\sqrt{3}}{2}) & \overline{\omega}'_{12} &= \frac{3-\sqrt{3}}{2}\,(i-1) \\
\overline{\omega}_{13} &= \tfrac{1}{2}\,(i-\sqrt{3}) & \overline{\omega}'_{13} &= i\,(2-\sqrt{3}) - 1
\end{aligned}\right. \\[2ex]
W_2 & \left\{\begin{aligned}
\overline{\omega}_{21} &= \frac{\sqrt{3}+1}{2}\,(i-1) & \overline{\omega}'_{21} &= i - 1 \\
\overline{\omega}_{22} &= -\,(1+i) & \overline{\omega}'_{22} &= 0 \\
\overline{\omega}_{23} &= i & \overline{\omega}'_{23} &= i - 1
\end{aligned}\right. \\[2ex]
W_3 & \left\{\begin{aligned}
\overline{\omega}_{31} &= \frac{\sqrt{3}+1}{2}\,(i-1) & \overline{\omega}'_{31} &= \frac{1-\sqrt{3}}{2} - 1 \\
\overline{\omega}_{32} &= \tfrac{1}{2} + i\,(1+\frac{\sqrt{3}}{2}) & \overline{\omega}'_{32} &= \frac{3+\sqrt{3}}{2}\,(i-1) \\
\overline{\omega}_{33} &= \tfrac{1}{2}\,(i+\sqrt{3}) & \overline{\omega}'_{33} &= i\,(2+\sqrt{3}) - 1
\end{aligned}\right.
\end{array}\right. \tag{88}$$

Form the following integrals

$$\begin{aligned}
u_1 &= W_1 + W_2 + W_3\,, \\
u_2 &= W_2\,, \\
u_3 &= \frac{W_1 - W_2}{\sqrt{3}}\,;
\end{aligned} \tag{89}$$

we can see from the above table that these will have very simple periods.


A3 +
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−
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A2 +
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a2
a1
A1
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A2 −
+
a3
a6
B3
+ −
a5
a4
− +
B2
−
A3 +

The table is

	A_1	A_2	A_3	B_1	B_2	B_3
u_1	$2i$	i	$2i$	$2i - 3$	$3(i-1)$	$5i - 3$
u_2	$1 + i$	$-1 - i$	i	$i - 1$	0	$i - 1$
u_3	$1 - i$	$-i$	-1	1	$1 - i$	$-2i.$

The integrals u_1, u_2, u_3, must be reducible to elliptic integrals, sin their periods are all expressible in the form

$$a + bi$$

where a and b are rational[1]). Here indeed they are integral.

The degree of the transformation which reduces the hyperellipti integral[1]) is when the a's and b's are integers

$$k = \sum_n (a_n b_{2+n} - b_n a_{3+n})$$

where the periods for the cross cuts A are

$$a_n + b_n i \qquad n = 1, 2, 3,$$

and for cross cuts B

$$a_{n+3} + b_{n+3} i \qquad n = 1, 2, 3.$$

We will now apply the Weierstrass-Picard theorem to the integral u_2.

According to this theorem, the periods of a hyperelliptic integral which is reducible, can by a canonical linear transformation be reduced to the following;

A_1	A_2	A_3	B_1	B_2	B_3
$k\omega$	o	o	ω'	ω	$o.$

The canonical linear transformation of the periods of u_2, which is necessary to reduce them to such a system, is

$$\begin{aligned}
\Omega_1 &= \bar{\omega}_1 + \bar{\omega}_3 - \bar{\omega}_1' + \bar{\omega}_2' - \bar{\omega}_3' \\
\Omega_2 &= \bar{\omega}_1' - \bar{\omega}_2' - \bar{\omega}_3' \\
\Omega_3 &= \bar{\omega}_2' \\
\Omega_1' &= \bar{\omega}_1' - \bar{\omega}_2' \\
\Omega_2' &= \bar{\omega}_3 - \bar{\omega}_1' + \bar{\omega}_2' \\
\Omega_3' &= -\bar{\omega}_1 - \bar{\omega}_2
\end{aligned}$$

Calling the paths at which the integral u_2 has the new periods (Ω_i) $L_1 L_2 L_3 L_4 L_5 L_6$, we obtain, by applying this linear transformation of periods to the integral $u_1 u_2 u_3$, the following table.

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[1]) See concerning the reduction problem Weierstrass Acta Math. 4, pag. 396; Picard, Bulletin de la S. M. F., Vol. 11 and 12; Poincaré, Bulletin de la S. M. F., Vol. 12, and American Journal, Vol. 8.

(94)
$$\begin{array}{lcccccc} & L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\ u_1 & 3 & -6i+3 & 3i-3 & -i & 3i & -3i \\ u_2 & 3 & 0 & 0 & -1+i & 1 & 0 \\ u_3 & 0 & 3i & 1-i & i & -1-i & --1+2i \end{array}$$

From these we obtain for the normal integrals $v_{i,}$ for the same paths

(95)
$$\begin{array}{lcccccc} & L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \\ v_1 & 1 & 0 & 0 & \frac{1}{3}(i-1) & \frac{1}{3} & 0 \\ v_2 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3}(i-1) & i \\ v_3 & 0 & 0 & 1 & 0 & i & 2i \end{array}$$

These integrals are again reducible, since their periods are expressible in the form $a + bi$ where a and b are rational.

The degree of the reduction is

$$k = \sum_n (a'_n b'_{3+n} + b'_n a'_{3+n})$$

where a''s and b''s denote the a's and b's for $3 v_1$, or $3 v_2$ or v_3 i. e. integers;

(96)
for v_1 $k = 3$
for v_2 $k = 6$
for v_3 $k = 2$.

The integrals themselves have the forms

$$v_1 = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{1}{\sqrt{\pi}} \int \frac{x dx}{\sqrt{x(1-x^6)}} \tag{97}$$

(Compare the reducible integral in Part III, § 1, where the reduction was of the third degree.)

$$v_2 = \frac{\sqrt{2\pi}(1-i)}{\Gamma(\frac{5}{12})\Gamma(\frac{1}{12})} \int \frac{(1-x^2)\,dx}{\sqrt{x(1-x^6)}} \tag{98}$$

$$v_3 = \frac{\sqrt{6\pi}(1+i)}{\Gamma(\frac{1}{12})\Gamma(\frac{5}{12})} e^{\frac{2\pi i}{3}} \int \frac{(x^2 - e^{\frac{5\pi i}{3}})\,dx}{\sqrt{x(1-x^6)}} \tag{99}$$

From examining the periods of $v_1 v_2 v_3$ we can easily see that any integral of the form

$$(q_1 + r_1 i)\, v_1 + (q_2 + r_2 i) v_2 + (q_3 + r_3 i)\, v_3$$

where the q's and r's are rational, must also have its periods expressible in the form $a + bi$ where a and b are rational and therefore all such integrals must be reducible.

As a particular case of

$$(q_1 + r_1 i)\, v_1 + (q_2 + r_2 i)\, v_2 + (q_3 + r_3 i)\, v_3 \tag{100}$$

we have $3\, v_2 - 2\, v_3$ as a reducible integral.